

Generalized Hahn-Banach Theorem on real linear spaces and Complex linear space.

Statement - Let L be a real linear space not necessarily normed and let P be a Sublinear functional on L i.e. P is a map from L into \mathbb{R} satisfying

$$P(\alpha x + \beta y) \leq P(\alpha x) + P(\beta y) \text{ for all } x, y \in L \text{ and all } \alpha, \beta \geq 0 \quad (1)$$

$$P(\alpha x) = \alpha P(x) \text{ for all } x \in L \text{ and all } \alpha \geq 0 \quad (2)$$

If f is a real linear functional defined on a linear subspace M such that $f(x) \leq P(x)$ for all $x \in M$, then there exists a real linear functional F defined on the whole L such that $f = F$ on M and $-F(x) \leq P(x)$ for all $x \in L$.

If L is complex linear space, then condition (1) is the same for but (2) is modified to $P(\alpha x) = |\alpha| P(x)$ for all $x \in L$ and all scalars α . And f is a complex linear functional on M such that $|f(x)| \leq P(x)$ for all $x \in M$.

The conclusion in this case is the same except that we have

$$|f(x)| \leq P(x) \quad \forall x \in L.$$

Proof - Let L be a real linear space.

If $x_0 \notin M$, consider the subspace

$$M_0 = (M \cup \{x_0\}) = \{x + \alpha x_0 : x \in M, \alpha \text{ real}\}$$

spanned by M and x_0 .

Define f_0 on M_0 by $f_0(x + \alpha x_0) = f(x) + \alpha r$

where r is any real number so that f_0 is real valued. It is easy to see that f_0 is linear on M_0

and $f_0 = f$ on M . If x_1, x_2 are any vectors in M ,

$$\text{then } f(x_2) - f(x_1) = f(x_2 - x_1) \leq P(x_2 - x_1) \text{ by hypothesis}$$

$$= P((x_2 + x_0) - (x_1 + x_0)) \leq P(x_2 + x_0)$$

$$+ P(-x_1 - x_0) \quad \text{--- by (1)}$$

so that $-f(x_1) - P(-x_1 - x_0) \leq -f(x_2) + P(x_2 + x_0)$

Since this inequality holds for arbitrary $x_1, x_2 \in M$

We conclude that

$$\sup_{y \in M} \{-f(y) - P(-y - x_0)\} \leq r_0 \leq \inf_{y \in M} \{-f(y) + P(y + x_0)\}$$

Choose r_0 to be any real number such that

$$\sup_{y \in M} \{-f(y) - P(-y - x_0)\} \leq r_0 \leq \inf_{y \in M} \{-f(y) + P(y + x_0)\}$$

It follows that

$-f(y) - P(-y - x_0) \leq r_0 \leq -f(y) + P(y + x_0)$ for all $y \in M$. With this choice of r_0 , we shall show that $f_0(x) \leq P(x)$ for all $x \in M_0$.

Let $w = x + \alpha x_0$ be an arbitrary element in M_0 . If $\alpha = 0$, then $f_0(w) = f(x) \leq P(x)$.

So let $\alpha \neq 0$ and put $y = x/\alpha$ in (3) to obtain

$$-f(x/\alpha) - P(-x/\alpha - x_0) \leq r_0 \leq -f(x/\alpha) + P(x/\alpha + x_0) \quad (4)$$

for all $x \in M$. If $\alpha > 0$, then right hand inequality in (4) gives

$$r_0 \leq -\frac{1}{\alpha} f(x) + \frac{1}{\alpha} P(x + \alpha x_0) \\ \Rightarrow f(x) + \alpha r_0 \leq P(x + \alpha x_0) \Rightarrow f_0(x + \alpha x_0) \leq P(x + \alpha x_0)$$

And if $\alpha < 0$, then the left hand inequality in (4) gives

$$r_0 \geq -f(x/\alpha) - P(-x/\alpha - x_0) = -f(x/\alpha) - P(-\frac{1}{\alpha}(x + \alpha x_0)) \\ = -\frac{1}{\alpha} f(x) - (-\frac{1}{\alpha}) P(x + \alpha x_0) \text{ by (1) since } -\frac{1}{\alpha} > 0 \\ = -\frac{1}{\alpha} f(x) + \frac{1}{\alpha} P(x + \alpha x_0)$$

We now multiply both sides of this inequality by α . Since $\alpha < 0$, the inequality will be reversed.

$$\therefore \alpha r_0 + f(x) \leq P(x + \alpha x_0) \Rightarrow f_0(x + \alpha x_0) \leq P(x + \alpha x_0)$$

Thus when $\alpha \neq 0$, we obtain

$$f_0(x + \alpha x_0) \leq P(x + \alpha x_0) \text{ for all } x \in M.$$

i.e. $f_0(w) \leq P(w)$ for all $w \in M_0$. Thus f_0 is a real linear functional on M_0 such that $f_0(x) = f(x)$ for all $x \in M$ and $f_0(w) \leq P(w)$ for all $w \in M_0$.

If $M_0 = L$, then we finish: if not we may repeat the process of extension but what guarantee is there that we shall ever extend to the whole space L .

It is at this point that Zorn's Lemma is needed.
 Let P denote the set of all ordered pairs (f_i, M_i)
 where f_i is an extension of f to the subspace
 $M_i \supset M$

And $f_i(x) \leq P(x)$ for all $x \in M_i$
 Partially order P by setting $(f_i, M_i) \leq (f_j, M_j)$ iff
 $M_i \subset M_j$ and $f_j = f_i$ on M_i . P is evidently non-
 empty. Let $\mathcal{Q} = \{f_i, M_i\}$ be a chain (i.e. a totally
 ordered set) in P . Then it is easy to see that \mathcal{Q}
 has an upper bound

$(\varphi, \cup M_i)$ where $\varphi(x) = f_i(x)$ for all $x \in M_i$.
 The point to be noted is $\cup M_i$ is a subspace of N
 and that φ is well-defined because of total
 ordering on \mathcal{Q} .

Hence by Zorn's Lemma, P contains a maximal
 element (F, H) . To complete the proof, we must
 show that $H = N$. Suppose, if possible, N contains
 H properly. Then there exists $x_0 \in N - H$ and by
 first part of the theorem, F can be extended
 to a functional F_0 on $H_0 = (H \cup \{x_0\})$ which
 contains H properly. But this contradicts the
 maximality of (F, H) . Consequently, we must have
 $H = N$ and the proof is complete.

ii. Let L be a complex linear space.

Here f is a complex linear functional
 on M such that $|f(x)| \leq P(x)$ for all $x \in M$.

Let $f_1 = \operatorname{Re} f$. Then $f_1(x) \leq |f(x)| \leq P(x)$ and so by case i,
 f_1 can be extended to a linear map F_1 of L into \mathbb{R}
 such that $F_1 = f_1$ on M and $F_1(x) \leq P(x)$ for all $x \in L$.

Define F by $F(x) = F_1(x) - i F_1(ix)$, $x \in L$.

Then it is easy to see that F is a linear functional
 on L such that $F = f$ on M . What remains to prove is
 that $|F(x)| \leq P(x)$ for all $x \in L$. Let $x \in L$ be arbitrary

and write $P(x) = r e^{i\theta}$ where $r \geq 0$ and θ is real. Then

$$|F(x)| = r = e^{-i\theta} \cdot r e^{i\theta} = F(e^{-i\theta} x) = F_1(e^{-i\theta} x) = F_1\left(\frac{e^{-i\theta} x}{r}\right) \cdot r$$

$\leq P\left(\frac{e^{-i\theta} x}{r}\right) \leq P(x)$ by (2) = $P(x)$. proved.

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